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Weighted estimates for singular integral operators and commutators associated with the sections ☆

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Abstract

In this paper we prove weighted estimates for singular integral operators and commutators associated with the sections.

Keywords: Singular integral operators; Commutators; Sections

1. Introduction

In 1996, Caffarelli and Gutiérez [2] studied real variable theory related to the Monge–Ampère equation. They defined the Hardy–Littlewood maximal operator M and $BMO_{\mathcal{F}}$ space associated to the sections and the doubling measure μ, and obtained the weak type $(1,1)$-boundedness of M and the John–Nirenberg inequality for $BMO_{\mathcal{F}}$ in [2]. In [3], Caffarelli and Gutiérez defined and proved L^2-boundedness of the singular integral operator H related to the Monge–Ampère equation. Later, Incognito [8] used the theory of homogeneous space to prove the weak type $(1,1)$ of H. The main purpose of this paper is to prove weighted estimates for the singular integral operator H and its commutator.

In this paper, we assume that the Borel measure μ satisfies the doubling condition (2.1). The main results of this paper can be stated as follows.

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Theorem 1.1. *Let H be a singular integral operator, $\omega(x)$ be a weight satisfying A_∞ condition* (*see their definitions in Section* 2) *and $0 < p < \infty$. Then the following a priori estimates hold*: *there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \left|Hf(x)\right|^p \omega(x)\,d\mu(x) \leqslant C \int_{\mathbb{R}^n} \left(Mf(x)\right)^p \omega(x)\,d\mu(x)$$

for any smooth function f for which the left-hand side is finite. Similarly, we have that there exists a constant $C > 0$ such that

$$\|Hf\|_{L^{p,\infty}(\omega)} \leqslant C\|Mf\|_{L^{p,\infty}(\omega)}$$

for any smooth function f for which the left-hand side is finite. Here $L^{p,\infty}(\omega)$ denotes the weak $L^p(\omega)$ space.

We remark that the first inequality in Theorem 1.1 was proved in the standard case by Coifman and Fefferman; see [5].

We also consider in this paper commutator of Coifman–Rochberg–Weiss $[b, H]$ defined by the formula

$$[b, H]f(x) = b(x)Hf(x) - H(bf)(x) = \int_{\mathbb{R}^n} \big(b(x) - b(y)\big)k(x, y)f(y)\,d\mu(y),$$

where $b \in BMO_{\mathcal{F}}$ defined in Section 2.

As in the case of singular integrals, we have

Theorem 1.2. *Let $b \in BMO_{\mathcal{F}}$, $0 < p < \infty$, $\omega(x)$ be a weight satisfying A_∞ condition. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \left|[b, H]f(x)\right|^p \omega(x)\,d\mu(x) \leqslant C \int_{\mathbb{R}^n} \left(M_{L\log L}f(x)\right)^p \omega(x)\,d\mu(x)$$

for any smooth function f for which the left-hand side is finite. Here $M_{L\log L}$ is defined in Section 2.

The weighted weak-type $(1, 1)$ estimate for the commutator is the following.

Theorem 1.3. *Let $b \in BMO_{\mathcal{F}}$. There exists a constant $C > 0$ such that for $\omega \in A_1$ and $\lambda > 0$*

$$\omega\left(\left\{x \in \mathbb{R}^n\colon \left|[b, H]f(x)\right| > \lambda\right\}\right) \leqslant C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\omega(x)\,d\mu(x).$$

We remark that the proofs of Theorem 1.2 and Theorem 1.3 follow some ideas from [11] and [12]. In the sequel, C is a positive constant which is independent of the main parameters and not necessary the same at each occurrence.

2. Preliminaries and notation

In this section we introduce some notation and basic tools needed for the proof of main results.

For $x \in \mathbb{R}^n$ and $t > 0$, let $S(x, t)$ denote open and bounded convex set containing x. We call $S(x, t)$ a section if the family $\{S(x, t)\colon x \in \mathbb{R}^n,\ t > 0\}$ is monotone increasing in t, i.e., $S(x, t) \subset S(x, t')$ for $t \leqslant t'$, and satisfies the following three conditions:

(A) There exist positive constants K_1, K_2, K_3 and ϵ_1, ϵ_2 such that given two sections $S(x_0,t_0)$, $S(x,t)$ with $t \leqslant t_0$ satisfying

$$S(x_0,t_0)\cap S(x,t)\neq\emptyset,$$

and an affine transformation T that "normalizes" $S(x_0,t_0)$, that is,

$$B(0,1/n)\subset T\bigl(S(x_0,t_0)\bigr)\subset B(0,1),$$

there exists $z\in B(0,K_3)$ depending on $S(x_0,t_0)$ and $S(x,t)$, which satisfies

$$B\bigl(z,K_2(t/t_0)^{\epsilon_2}\bigr)\subset T\bigl(S(x,t)\bigr)\subset B\bigl(z,K_1(t/t_0)^{\epsilon_1}\bigr),$$

and

$$T(x)\in B\bigl(z,(1/2)K_2(t/t_0)^{\epsilon_2}\bigr).$$

Here and below $B(x,t)$ denotes the Euclidean ball centered at x with radius t.

(B) There exists a constant $\delta>0$ such that given a section $S(x,t)$ and $y\notin S(x,t)$, if T is an affine transformation that "normalizes" $S(x,t)$, then for any $0<\epsilon<1$

$$B\bigl(T(y),\epsilon^{\delta}\bigr)\cap T\bigl(S\bigl(x,(1-\epsilon)t\bigr)\bigr)=\emptyset.$$

(C) $$\bigcap_{t>0}S(x,t)=\{x\}\quad\text{and}\quad\bigcup_{t>0}S(x,t)=\mathbb{R}^n.$$

In addition, we also assume that a Borel measure μ which is finite on compact sets is given, $\mu(\mathbb{R}^n)=\infty$, and satisfies the following doubling property with respect to $\mathcal{F}$, where $\mathcal{F}=\{S(x,t)\colon\ x\in\mathbb{R}^n,\ t>0\}$, that is, there exists a constant A such that

$$\mu\bigl(S(x,2t)\bigr)\leqslant A\mu\bigl(S(x,t)\bigr)\quad\text{for any section } S(x,t)\in\mathcal{F}. \tag{2.1}$$

Properties (A) and (B) of the sections imply the following engulfing property proved in [1]; there exists a constant $\theta>1$, depending only on K_1 and ϵ_1, such that for $y\in S(x,r)$ we have

(D) $S(y,r)\subset S(x,\theta r)$ and $S(x,r)\subset S(y,\theta r)$.

We use the sections to define a pseudo-distance function: $\rho(x,y)=\inf\{t>0\colon\ y\in S(x,t)\}$. The engulfing property of the sections implies the following two properties of ρ:

$$\rho(x,y)\leqslant\theta\rho(y,x), \tag{2.2}$$

and

$$\rho(x,y)\leqslant\theta^2\bigl(\rho(x,z)+\rho(z,y)\bigr), \tag{2.3}$$

see [8]. It is easy to see that

(E) $\rho(x,y)\leqslant d(x,y)\leqslant\theta\rho(x,y)$.

In [1], the authors proved that if a family $\mathcal{F}$ of sections satisfies the properties (A), (B) and (C), then there exists a quasi-metric $d(x,y)$ on $\mathbb{R}^n$ with respect to $\mathcal{F}$ defined by

$$d(x,y)=\inf\bigl\{r\colon\ x\in S(y,r)\text{ and }y\in S(x,r)\bigr\}.$$

The triangular constant of the quasi-metric d is just the θ appeared in the property (D), that is,

$$d(x,y)\leqslant\theta\bigl(d(x,z)+d(z,y)\bigr)\quad\text{for any } x,y,z\in\mathbb{R}^n.$$

Moreover, denoting by $B_d(x,r)=\{y\in\mathbb{R}^n\colon\ d(x,y)<r\}$ the d-ball centered at x with radius r.

The following relationship between a section and a d-ball can be found in [1].

(F) For any $x\in\mathbb{R}^n$ and any $r>0$, $S(x,r/2\theta)\subset B_d(x,r)\subset S(x,r)$.

An important example coming from the Monge–Ampère equation is where we let ϕ be a smooth solution whose graph contains no lines. Then we let $\rho(x,y)=\phi(y)-\phi(x)-\nabla\phi(x)\times(y-x)$, and define the sections by $S(x,t)=\{y\colon\ \rho(x,y)<t\}$. These sections satisfy the properties (A), (B) and (C); see [2].

We shall consider kernels $k(x,y)$ that can be represented in the form

$$k(x,y)=\sum_i k_i(x,y), \tag{2.4}$$

where the k_i's satisfy the following properties:

$$\operatorname{supp} k_i(\cdot,y)\subset S_i(y),\quad \forall y; \tag{2.5}$$

$$\operatorname{supp} k_i(x,\cdot)\subset S_i(x),\quad \forall y; \tag{2.6}$$

$$\int_{\mathbb{R}^n}\bigl|k_i(x,y)\bigr|\,d\mu(y)=\int_{\mathbb{R}^n}k_i(x,y)\,d\mu(x)=0,\quad \forall x,y; \tag{2.7}$$

$$\sup_i\int_{\mathbb{R}^n}\bigl|k_i(x,y)\bigr|\,d\mu(y)\leqslant C_1,\quad \forall x; \tag{2.8}$$

$$\sup_i\int_{\mathbb{R}^n}\bigl|k_i(x,y)\bigr|\,d\mu(x)\leqslant C_1,\quad \forall y; \tag{2.9}$$

where $S_i(x)=S(x,2^i)$ for any $x\in\mathbb{R}^n$ and $i\in\mathbb{Z}$, if T is an affine transformation that normalizes the section $S_i(y)$ then k_i satisfies the Lipschitz condition

$$\bigl|k_i(u,y)-k_i(v,y)\bigr|\leqslant C_2\frac{1}{\mu(S_i(y))}|Tu-Tv|; \tag{2.10}$$

and finally, if T is an affine transformation that normalizes the section $S_i(x)$ then k_i satisfies the Lipschitz condition

$$\bigl|k_i(x,u)-k_i(x,v)\bigr|\leqslant C_2\frac{1}{\mu(S_i(x))}|Tu-Tv|. \tag{2.11}$$

The operator associated to the kernel k is defined by

$$Hf(x)=\int_{\mathbb{R}^n}k(x,y)f(y)\,d\mu(y).$$

It was proved in [2] that the operator H is the strong-type $(2,2)$. Subsequently, the weak-type $(1,1)$ of H was proved in [8].

We say that b is $BMO_{\mathcal{F}}$ function, that is,

$$\|b\|_*:=\sup_{S\in\mathcal{F}}\frac{1}{\mu(S)}\int_S\bigl|b(x)-m_S(b)\bigr|\,d\mu(x)<\infty,$$

where $m_S(b)=\frac{1}{\mu(S)}\int_S b(x)\,d\mu(x)$.

It should be pointed out that the quasi-metric d and the Borel measure μ satisfying the doubling condition (2.1) create a space of homogeneous type; see also [1,3].

A weight will always mean a positive function which is locally integrable. We say that a weight ω belongs to the class A_p for $1 < p < \infty$, if there is a constant C such that for all ball $B = B_d(x,r)$

$$\left(\frac{1}{\mu(B)}\int_B \omega(y)\,d\mu(y)\right)\left(\frac{1}{\mu(B)}\int_B \omega^{-\frac{1}{p-1}}(y)\,d\mu(y)\right)^{p-1} \leqslant C.$$

We also say that a nonnegative function ω satisfies the A_1 condition if there exists a constant C for all balls B

$$\frac{1}{\mu(B)}\int_B \omega(y)\,d\mu(y) \leqslant C \inf_{x\in B} \omega(x).$$

We define $A_\infty(\mathbb{R}^n) = \bigcup_{p\geqslant 1} A_p(\mathbb{R}^n)$.

Standard real analysis tools as the maximal function Mf, the sharp function $M^\sharp f$, the *BMO* space, naturally carries over to this context, namely,

$$Mf(x) = \sup_{x\in B} \frac{1}{\mu(B)}\int_B \left|f(y)\right|d\mu(y),$$

$$M^\sharp f(x) = \sup_{x\in B} \frac{1}{\mu(B)}\int_B \left|f(y) - f_B\right|d\mu(y) \approx \sup_{x\in B}\inf_C \frac{1}{\mu(B)}\int_B \left|f(y) - C\right|d\mu(y),$$

where $f_B = \frac{1}{\mu(B)}\int_B f(y)\,d\mu(y)$. And

$$\|f\|_{BMO} = \sup_{x\in B} \frac{1}{\mu(B)}\int_B \left|f(y) - f_B\right|d\mu(y) < \infty.$$

It is easy to see that the $BMO_{\mathcal{F}}$ space coincides with the *BMO* space and

$$\|f\|_* \approx \|f\|_{BMO}.$$

A variant of maximal operator and sharp maximal operator $M_\delta f(x) = M(|f|^\delta)^{1/\delta}(x)$ and $M_\delta^\sharp f(x) = M^\sharp(|f|^\delta)^{1/\delta}(x)$, which will become the main tool in our scheme.

The main inequality between these operators to be used is a version of the homogeneous spaces due to [9,10].

Theorem 2.1. *Let $0 < p, \delta < \infty$ and $\omega \in A_\infty$. There exists a positive C such that*

$$\int_{\mathbb{R}^n} M_\delta f(x)^p \omega(x)\,d\mu(x) \leqslant C \int_{\mathbb{R}^n} M_\delta^\sharp f(x)^p \omega(x)\,d\mu(x)$$

for any smooth function f for which the left-hand side is finite.

We next recall some basic definitions and facts about Orlicz spaces, referring to [13] for a complete account.

A function $\Phi : [0,\infty) \to [0,\infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. If Φ is a Young function, we define the Φ-average of a function f over a ball B by means of the following Luxemberg norm:

$$\|f\|_{\Phi,B} = \inf\left\{\lambda > 0:\ \frac{1}{\mu(B)}\int_B \Phi\left(\frac{|f|}{\lambda}\right)d\mu \leqslant 1\right\}.$$

The generalized Hölder's inequality

$$\frac{1}{\mu(B)}\int_B |fg|\,d\mu \leqslant \|f\|_{\Phi,B}\|g\|_{\bar{\Phi},B} \tag{2.12}$$

holds, where $\bar{\Phi}$ is the complementary Young function associated to Φ. And we define the corresponding maximal function

$$M_\Phi f(x) = \sup_{B:\ x\in B} \|f\|_{\Phi,B}. \tag{2.13}$$

Then in example that we are going to use is $\Phi(t) = t(1+\log^+ t)$ with the maximal function denoted by $M_{L\log L}$. The complementary Young function is given by $\bar{\Phi}(t) \approx e^t$ with the corresponding maximal function denoted by $M_{\exp L}$.

3. Pointwise estimates

In this section we prove the basic pointwise estimates for the singular integral operators and their commutators related to the sections.

Lemma 3.1. *Let $k(x,y) = \sum_i k_i(x,y)$. Then there exists a positive constant C such that*

$$\left|k(x,y_0) - k(x,y)\right| + \left|k(y_0,x) - k(y,x)\right| \leqslant \frac{C}{\mu(S(y_0, 2^k\rho(y_0,y)))}2^{-\epsilon_1 k},$$

if $\rho(y_0,x) \geqslant 2^k 4\theta^2\rho(y_0,y)$ and any nonnegative k.

Proof. By the symmetry of kernel $k(x,y)$, we only need to prove that

$$\left|k(x,y_0) - k(x,y)\right| \leqslant \frac{C}{\mu(S(y_0, 2^k\rho(y_0,y)))}2^{-\epsilon_1 k}, \tag{3.1}$$

if $\rho(y_0,x) \geqslant 2^k 4\theta^2\rho(y_0,y)$ and $k \geqslant 0$.

Let i_0 be such that $2^{i_0-1} \leqslant \rho(y_0,y) \leqslant 2^{i_0}$. Then, from the definition of $k(x,y)$, we have

$$\begin{aligned}
\left|k(x,y_0) - k(x,y)\right| &= \left|\sum_i \bigl(k_i(x,y_0) - k_i(x,y)\bigr)\right| \\
&\leqslant \sum_i \left|k_i(x,y_0) - k_i(x,y)\right| \\
&= \sum_{i>i_0+k} \left|k_i(x,y_0) - k_i(x,y)\right|.
\end{aligned} \tag{3.2}$$

The last equality follows from the fact if $i \leqslant i_0 + k$ then $\rho(y_0,x) \geqslant 2^k 4\theta^2\rho(y_0,y)$ implies that $x \notin S_i(y_0) \cup S_i(y)$ and hence by (2.5), $k_i(x,y_0) - k_i(x,y) = 0$ for $i \leqslant i_0 + k$. Let T_i normalize $S_i(x)$, then by (2.11), we have

$$\left|k_i(x,y_0) - k_i(x,y)\right| \leqslant \frac{C}{\mu(S_i(x))}\left|T_i(y_0) - T_i(y)\right|. \tag{3.3}$$

First, we estimate $|T_i(y_0) - T_i(y)|$. We can assume

$$S_i(x) \cap S_{i_0}(y_0) \neq \emptyset.$$

Since otherwise $k_i(x, y_0)$ and $k_i(x, y)$ are both zero by (2.6). Thus, we have

$$T_i\big(S_{i_0}(y_0)\big) \subset B\big(z, K_1 2^{\epsilon_1(i_0-i)}\big)$$

by property (A) of the sections. Since $y \in S_{i_0}(y_0)$,

$$\big|T_i(y_0) - T_i(y)\big| \leqslant 2K_1 2^{\epsilon_1(i_0-i)}. \tag{3.4}$$

Since $x \in S_i(y_0) \cup S_i(y)$, if $x \in S_i(y_0)$, that is, $\rho(y_0, x) \leqslant 2^i$, then we have

$$S_i(x) \subset S\big(y_0, \theta^2 2^{i+1}\big). \tag{3.5}$$

If $x \in S_i(y_0)$, that is, $\rho(y, x) \leqslant 2^i$, so $\rho(y_0, x) \leqslant \theta^2(\rho(y_0, y) + \rho(y, x)) \leqslant \theta^2 2^{i+1}$, then we have

$$S_i(x) \subset S\big(y_0, \theta^4 2^{i+2}\big). \tag{3.6}$$

Hence, from (3.5) and (3.6), when $x \in S_i(y_0) \cup S_i(y)$, we have

$$S_i(x) \subset S\big(y_0, \theta^4 2^{i+2}\big). \tag{3.7}$$

Putting (3.3), (3.4) and (3.7) into (3.2), by the doubling property, we obtain

$$\begin{aligned}
\sum_{i>i_0+k} \big|k_i(x, y_0) - k_i(x, y)\big| &\leqslant \sum_{i>i_0+k} \frac{1}{\mu(S_i(x))} 2k_1 2^{\epsilon_1(i_0-i)} \\
&\leqslant \frac{C}{\mu(S_{i_0+k}(y_0))} \sum_{i>i_0+k} 2^{\epsilon_1(i_0-i)} \\
&\leqslant \frac{C}{\mu(S_{i_0+k}(y_0))} 2^{-\epsilon_1 k}.
\end{aligned}$$

Thus, (3.1) holds. Lemma 3.1 is proved. $\square$

Lemma 3.2. *Let* $0 < \delta < 1$. *Then there exists a constant* $C > 0$ *such that*

$$M_\delta^\sharp(Hf)(x) \leqslant CM(f)(x) \tag{3.8}$$

for any smooth function f *and every* $x \in \mathbb{R}^n$.

Proof. Fix $x \in \mathbb{R}^n$ and let $B = B_d(x, r)$. Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\bar{B}}$, where $\bar{B} = B_d(x, 16\theta^4 r)$. Set $C = |(Hf_2)_B|$.

Since $0 < \delta < 1$, we can estimate

$$\begin{aligned}
&\left(\frac{1}{\mu(B)} \int_B \big|\big|Hf(y)\big|^\delta - C^\delta\big|\, d\mu(y)\right)^{1/\delta} \\
&\quad\leqslant \left(\frac{1}{\mu(B)} \int_B \big|\big|Hf(y)\big| - \big|(Hf_2)_B\big|\big|^\delta\, d\mu(y)\right)^{1/\delta} \\
&\quad\leqslant \left(\frac{1}{\mu(B)} \int_B \big|Hf(y) - (Hf_2)_B\big|^\delta\, d\mu(y)\right)^{1/\delta}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left(\frac{1}{\mu(B)}\int_B \left|Hf_1(y)\right|^\delta d\mu(y)\right)^{1/\delta} + C\left(\frac{1}{\mu(B)}\int_B \left|Hf_2(y)-(Hf_2)_B\right|^\delta d\mu(y)\right)^{1/\delta}\\
&= I + II.
\end{aligned}
$$

For I, we recall that H is weak type $(1,1)$. Then by Kolmogorov's inequality (see [14]), we have

$$
\begin{aligned}
I &\leqslant \frac{C}{\mu(B)}\|Hf_1\|_{L^{1,\infty}}\\
&\leqslant \frac{C}{\mu(\bar{B})}\int_{\bar{B}} \left|f(y)\right| d\mu(y)\\
&\leqslant CMf(x). \qquad (3.9)
\end{aligned}
$$

To estimate II we will use the definition of H and Lemma 3.1 to obtain the following

$$
\begin{aligned}
II &\leqslant \frac{C}{\mu(B)}\int_B \left|Hf_2(y)-(Hf_2)_B\right| d\mu(y)\\
&\leqslant \frac{C}{\mu(B)^2}\int_B\int_B\int_{\mathbb{R}^n\setminus\bar{B}} \left|k(y,\omega)-k(z,\omega)\right|\left|f(\omega)\right| d\mu(\omega)\,d\mu(z)\,d\mu(y)\\
&\leqslant \frac{C}{\mu(B)^2}\int_B\int_B\int_{d(x,\omega)>16\theta^4 r} \left|k(y,\omega)-k(z,\omega)\right|\left|f(\omega)\right| d\mu(\omega)\,d\mu(z)\,d\mu(y)\\
&\leqslant \frac{C}{\mu(B)^2}\int_B\int_B\int_{\rho(x,\omega)>16\theta^5 r} \left|k(y,\omega)-k(z,\omega)\right|\left|f(\omega)\right| d\mu(\omega)\,d\mu(z)\,d\mu(y)\\
&\leqslant \frac{C}{\mu(B)^2}\int_B\int_B\sum_{k=1}^{\infty}\int_{2^k r_1\leqslant\rho(x,\omega)<2^{k+1}r_1} \left|k(y,\omega)-k(z,\omega)\right|\left|f(\omega)\right| d\mu(\omega)\,d\mu(z)\,d\mu(y)\\
&\leqslant C\sum_{k=1}^{\infty}\frac{2^{-\epsilon_1 k}}{\mu(S(x,2^k r_1))}\int_{S(x,2^{k+1}r_1)} \left|f(\omega)\right| d\mu(\omega)\\
&\leqslant CMf(x), \qquad (3.10)
\end{aligned}
$$

where $r_1 = 16\theta^5 r$. □

Next we need a similar estimate for the commutator.

Lemma 3.3. *Let $h \in BMO$ and let $0<\delta<\epsilon<1$. Then there exists a constant $C>0$ such that*

$$M_\delta^\sharp\big([b,H]f\big)(x) \leqslant C\|b\|_{BMO}\big(M_\epsilon(Hf)(x) + M_{L\log L}(f)(x)\big) \qquad (3.11)$$

and for $r>1$

$$M_\delta^\sharp\big([b,H]f\big)(x) \leqslant C\|b\|_{BMO}\big(M_{\delta r}(Hf)(x) + M_r(f)(x)\big) \qquad (3.12)$$

for any smooth function f and every $x \in \mathbb{R}^n$.

Proof. We first estimate (3.11). Observe that for any constant λ

$$[b,H]f(x)=\big(h(x)-\lambda\big)Hf(x)-H\big((b-\lambda)f\big)(x).$$

As above we fix $x\in\mathbb{R}^n$ and let $B=B_d(x,r)$. Decompose $f=f_1+f_2$, where $f_1=f\chi_{\bar B}$, where $\bar B=B_d(x,16\theta^4 r)$. Let λ be a constant and C a constant to be fixed along the proof.

Since $0<\delta<1$, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B \big||[b,H]f(y)|^\delta-|C|^\delta\big|\,d\mu(y)\right)^{1/\delta}\\
&\quad\leqslant\left(\frac{1}{\mu(B)}\int_B \big|[b,H]f(y)-C\big|^\delta\,d\mu(y)\right)^{1/\delta}\\
&\quad\leqslant\left(\frac{1}{\mu(B)}\int_B \big|\big(h(y)-\lambda\big)Hf(y)-H\big((b-\lambda)f\big)(y)-C\big|^\delta\,d\mu(y)\right)^{1/\delta}\\
&\quad\leqslant C\left(\frac{1}{\mu(B)}\int_B \big|\big(h(y)-\lambda\big)Hf(y)\big|^\delta\,d\mu(y)\right)^{1/\delta}\\
&\qquad+C\left(\frac{1}{\mu(B)}\int_B \big|H\big((b-\lambda)f_1\big)(y)\big|^\delta\,d\mu(y)\right)^{1/\delta}\\
&\qquad+C\left(\frac{1}{\mu(B)}\int_B \big|H\big((b-\lambda)f_2\big)(y)-C\big|^\delta\,d\mu(y)\right)^{1/\delta}\\
&\quad=I+II+III.
\end{aligned}$$

To deal with I, we first fix $\lambda=b_{\bar B}$, the average of b on $\bar B$. Then for any $1<p<\epsilon/\alpha$, by the John–Nirenberg inequality of $BMO_{\mathcal F}$ (see [3,7]), we have

$$\begin{aligned}
I&=C\left(\frac{1}{\mu(B)}\int_B \big|\big(b(y)-b_{\bar B}\big)Hf(y)\big|^\delta\,d\mu(y)\right)^{1/\delta}\\
&\leqslant C\left(\frac{1}{\mu(\bar B)}\int_{\bar B} \big|b(y)-b_{\bar B}\big|^{p'\delta}\,d\mu(y)\right)^{1/p'\delta}\left(\frac{1}{\mu(B)}\int_B \big|Hf(y)\big|^{p\delta}\,d\mu(y)\right)^{1/p\delta}\\
&\leqslant C\|b\|_{BMO}M_{\delta p}(Hf)(x)\\
&\leqslant C\|b\|_{BMO}M_{\epsilon}(Hf)(x),
\end{aligned}\tag{3.13}$$

where $1/p'+1/p=1$.

For II, we make use of Kolmogorov's inequality again, then

$$\begin{aligned}
II&\leqslant C\frac{1}{\mu(B)}\int_B \big|\big((b-b_{\bar B})f_1\big)(y)\big|\,d\mu(y)\\
&\leqslant C\frac{1}{\mu(\bar B)}\int_B \big|b(y)-b_{\bar B}\big|\big|f(y)\big|\,d\mu(y)\\
&\leqslant C\|b-b_{\bar B}\|_{\exp L,\bar B}\|f\|_{L\log L,\bar B}\\
&\leqslant C\|b\|_{BMO}M_{L\log L}(f)(x),
\end{aligned}\tag{3.14}$$

where we have used (2.12) and (2.13).

Finally, for III we first fix the value of C by taking $C = (H((b - b_{\bar B}) f_2))_B$, the average of $H((b - b_{\bar B}) f_2)$ on B. Then, we have

$$\begin{aligned}
III &\leqslant \frac{C}{\mu(B)} \int_B \big|H\big((b - b_{\bar B}) f_2\big)(y) - \big(H\big((b - b_{\bar B}) f_2\big)\big)_B\big|\, d\mu(y) \\
&\leqslant \frac{C}{\mu(B)^2} \int_B \int_B \int_{\mathbb{R}^n \setminus \bar B} \big|k(y, \omega) - k(z, \omega)\big|\big|\big(b(\omega) - b_{\bar B}\big) f(\omega)\big|\, d\mu(\omega)\, d\mu(z)\, d\mu(y) \\
&\leqslant \frac{C}{\mu(B)^2} \int_B \int_B \int_{d(x,\omega) > 16\theta^4 r} \big|k(y, \omega) - k(z, \omega)\big|\big|\big(b(\omega) - b_{\bar B}\big) f(\omega)\big|\, d\mu(\omega)\, d\mu(z)\, d\mu(y) \\
&\leqslant \frac{C}{\mu(B)^2} \int_B \int_B \sum_{k=1}^{\infty} \int_{2^k r_1 \leqslant \rho(x,\omega) < 2^{k+1} r_1} \big|k(y, \omega) - k(z, \omega)\big| \\
&\quad \times \big|\big(b(\omega) - b_{\bar B}\big) f(\omega)\big|\, d\mu(\omega)\, d\mu(z)\, d\mu(y) \\
&\leqslant C \sum_{k=1}^{\infty} \frac{2^{-\epsilon_1 k}}{\mu(B_d(x, 2^{k+1} r_1))} \int_{B_d(x, 2^{k+1} r_1)} \big|b(\omega) - b_{\bar B}\big|\big|f(\omega)\big|\, d\mu(\omega) \\
&\leqslant C \sum_{k=1}^{\infty} \frac{2^{-\epsilon_1 k}}{\mu(B_d(x, 2^{k+1} r_1))} \int_{B_d(x, 2^{k+1} r_1)} \big|b(\omega) - b_{B_d(x, 2^{k+1} r_1)}\big|\big|f(\omega)\big|\, d\mu(\omega) \\
&\quad + C \sum_{k=1}^{\infty} \frac{2^{-\epsilon_1 k}}{\mu(B_d(x, 2^{k+1} r_1))} \big|b(\bar B) - b_{B_d(x, 2^{k+1} r_1)}\big| \int_{B_d(x, 2^{k+1} r_1)} \big|f(\omega)\big|\, d\mu(\omega) \\
&\leqslant C \sum_{k=1}^{\infty} 2^{-\epsilon_1 k} \|b - b_{B_d(x, 2^{k+1} r_1)}\|_{\exp L, B_d(x, 2^{k+1} r_1)} \|f\|_{L \log L, B_d(x, 2^{k+1} r_1)} \\
&\quad + C \|b\|_{BMO} M(f)(x) \sum_{k=1}^{\infty} k 2^{-\epsilon_1 k} \\
&\leqslant C \|b\|_{BMO} M_{L \log L}(f)(x),
\end{aligned} \tag{3.15}$$

where $r_1 = 16\theta^5 r$ and in last inequality we have used that $|b_{\bar B} - b_{B_d(x, 2^{k+1} r_1)}| \leqslant Ck\|b\|_{BMO}$ and $M(f)(x) \leqslant M_{L \log L}(f)(x)$.

From (3.12), (3.14) and (3.15) we get (3.11). The proof of (3.12) is similar to that of (3.11), we omit the details here. Hence the proof is finished. □

4. Proof of the main theorems

Proof of Theorem 1.1. Since $\omega \in A_\infty$, we can combine Theorem 2.1 in Section 2 together with Lemma 3.2 with $0 < \delta < 1$ to get

$$\begin{aligned}
\int_{\mathbb{R}^n} \big|Hf(x)\big|^p \omega(x)\, d\mu(x) &\leqslant \int_{\mathbb{R}^n} \big(M_\delta(Hf)(x)\big)^p \omega(x)\, d\mu(x) \\
&\leqslant C \int_{\mathbb{R}^n} \big(M_\delta^\sharp(Hf)(x)\big)^p \omega(x)\, d\mu(x)
\end{aligned}$$

$$\leqslant C \int_{\mathbb{R}^n} \big(M(f)(x)\big)^p \omega(x)\, d\mu(x).$$

The second part of Theorem 1.1 will be deduced from the first part of Theorem 1.1 and Theorem 1.1 in [4]. □

Proof of Theorem 1.2. From Theorem 2.1 in Section 2, since $\omega \in A_\infty$, we can combine Lemma 3.2 together with Lemma 3.3 with $0 < \delta < \epsilon < 1$ to get

$$\begin{aligned}
\big\|[b,H]f(x)\big\|_{L^p(\omega)} &\leqslant \big\|M_\delta\big([b,H]f\big)\big\|_{L^p(\omega)} \\
&\leqslant \big\|M_\delta^\sharp\big([b,H]f\big)\big\|_{L^p(\omega)} \\
&\leqslant C\|b\|_{BMO}\big(\big\|M_\epsilon(Hf)\big\|_{L^p(\omega)} + \big\|M_{L\log L}(f)\big\|_{L^p(\omega)}\big) \\
&\leqslant C\|b\|_{BMO}\big(\big\|M_\epsilon^\sharp(Hf)\big\|_{L^p(\omega)} + \big\|M_{L\log L}(f)\big\|_{L^p(\omega)}\big) \\
&\leqslant C\|b\|_{BMO}\big(\|Mf\|_{L^p(\omega)} + \big\|M_{L\log L}(f)\big\|_{L^p(\omega)}\big) \\
&\leqslant C\|b\|_{BMO}\big\|M_{L\log L}(f)\big\|_{L^p(\omega)}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.3. Without loss of generality, we can assume that $\|b\|_{BMO_{\mathcal{F}}} = 1$. We know that $\omega \in A_1 \subset A_p$ for every $1 < p < \infty$, then from (3.12) in Lemma 3.2, we know that $[b,H]$ is bounded on $L^p(\omega)$. As in [6], we can perform the Calderón–Zygmund decomposition at the level $\lambda > 0$ and there exists a collection of balls $\{B_j\}$ and $\{x \in \mathbb{R}^n\colon\ Mf(x) > C_\mu\lambda\} = \bigcup_j B_j = \bigcup_j B_d(x_j, r_j)$ where C_μ is a positive constant depending only the measure μ such that

- (i) $|f(x)| \leqslant C\lambda$, for μ a.e., $x \in \mathbb{R}^n \setminus \bigcup_j B_j$,
- (ii) $\frac{1}{\mu(B_j)}\int_{B_j} |f(y)|\, d\mu(y) \leqslant C\lambda$,
- (iii) $\sum_j \mu(B_j) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(y)|\, d\mu(y)$,
- (iv) there exists an integer $N \geqslant 1$, independent of f and λ, such that, every point in $\mathbb{R}^n$ belongs at most to N of the balls.

Similarly, we decompose f as $f = g + h = g + \sum_j h_j$, where

$$g(x) = f(x)\chi_{\mathbb{R}^n \setminus \bigcup_j B_j} + \sum_j \left(\frac{1}{\mu(B_j)}\int_{B_j} f(y)\rho_j(y)\, d\mu(y)\right)\chi_{B_j}(x),$$

$$h_j(x) = f(x)\rho_j(x) - \left(\frac{1}{\mu(B_j)}\int_{B_j} f(y)\rho_j(y)\, d\mu(y)\right)\chi_{B_j}(x),$$

$$\rho_j(x) = \frac{\chi_{B_j}(x)}{\sum_j \chi_{B_j}(x)}\chi_{\bigcup_j B_j}.$$

Let $\bar{\Omega} = \bigcup_j \bar{B}_j$, where $\bar{B}_j = B_d(z_j, 4\theta^3 r_j)$. Then

$$\begin{aligned}
\omega\big(\big\{y \in \mathbb{R}^n\colon\ \big|[b,H]f(y)\big| > \lambda\big\}\big) &\leqslant \omega\big(\big\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon\ \big|[b,H]g(y)\big| > \lambda/2\big\}\big) + \omega(\bar{\Omega}) \\
&\quad + \omega\big(\big\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon\ \big|[b,H]h(y)\big| > \lambda/2\big\}\big). \qquad (4.1)
\end{aligned}$$

Since $A_1 \subset A_2$, it follows that $[b, H]$ is a bounded operator on $L^2(\omega)$. Note that $|g(x)| \leqslant C\lambda$ for μ almost all $x \in \mathbb{R}^n$, then

$$\begin{aligned}
&\omega\big(\big\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon \big|[b, H]g(y)\big| > \lambda/2\big\}\big) \\
&\quad\leqslant \frac{C}{\lambda^2}\int_{\mathbb{R}^n} \big|g(x)\big|^2 \omega(x)\,d\mu(x) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} \big|g(x)\big|\omega(x)\,d\mu(x) \\
&\quad\leqslant \frac{C}{\lambda}\|f\|_{L^1(\omega)} + \sum_j \frac{1}{\mu(B_j)}\int_{B_j}\omega(x)\,d\mu(x)\int_{B_j}\big|f(y)\big|\,d\mu(y) \\
&\quad\leqslant \frac{C}{\lambda}\|f\|_{L^1(\omega)}.
\end{aligned} \tag{4.2}$$

For the second term of (4.1), by (iv) and the weak type (1, 1) of M, we get

$$\begin{aligned}
\omega(\bar{\Omega}) &\leqslant \sum_j \omega(\bar{B}_j) \leqslant C\sum_j \omega(B_j) \\
&= C\int_{\mathbb{R}^n}\sum_j \chi_{B_j}(x)\omega(x)\,d\mu(x) \\
&\leqslant CN\omega\Big(\bigcup_j B_j\Big) \\
&= CN\omega\big(\big\{x \in \mathbb{R}^n\colon Mf(x) > C_\mu\lambda\big\}\big) \\
&\leqslant \frac{C}{\lambda}\|f\|_{L^1(\omega)}.
\end{aligned} \tag{4.3}$$

Finally, for the third term of (4.1), we have

$$\begin{aligned}
&\omega\big(\big\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon \big|[b, H]h(y)\big| > \lambda/2\big\}\big) \\
&\quad\leqslant \omega\bigg(\bigg\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon \sum_j \big|b(x) - b_{B_j}\big|\big|H(h_j)(x)\big| > \lambda/4\bigg\}\bigg) \\
&\qquad + \omega\bigg(\bigg\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon \bigg|H\bigg(\sum_j \big(b(x) - b_{B_j}\big)h_j\bigg)(x)\bigg| > \lambda/4\bigg\}\bigg) \\
&\quad:= I + II.
\end{aligned}$$

To estimate I, using Lemma 3.1 and the cancellation of h_j over $B_j = B_d(z_j, r_j)$, let $2^k B_j = B_d(z_j, 2^k 16\theta^4 r_j)$, by the Hölder inequality and the John–Nirenberg inequality of $BMO_{\mathcal{F}}$ (see [3,7]), we have

$$\begin{aligned}
I &\leqslant \frac{C}{\lambda}\sum_j \int_{\mathbb{R}^n\setminus\bar{\Omega}} \big|b(x) - b_{B_j}\big|\big|H(h_j)(x)\big|\omega(x)\,d\mu(x) \\
&\leqslant \frac{C}{\lambda}\sum_j \int_{\mathbb{R}^n\setminus\bar{\Omega}} \big|b(x) - b_{B_j}\big|\int_{B_j}\big|k(x,y) - k(x,z_j)\big|\big|h_j(y)\big|\,d\mu(y)\,\omega(x)\,d\mu(x) \\
&\quad + \frac{C}{\lambda}\sum_j \int_{B_j}\big|h_j(y)\big|\,d\mu(y)\sum_{k=1}^{\infty}\frac{2^{-\epsilon_1 k}}{\mu(S(2^k B_j))}\int_{2^k B_j}\big|b(x) - b_{B_j}\big|\omega(x)\,d\mu(x)
\end{aligned}$$

$$\leqslant \frac{C}{\lambda}\sum_j \int_{B_j} \left|h_j(y)\right| \sum_{k=1}^{\infty} \frac{2^{-\epsilon_1 k}}{\mu(2^k B_j)} \int_{2^k B_j} \left|b(x)-b_{2^{k+1}B_j}\right|\omega(x)\,d\mu(x)\,d\mu(y)$$
$$+\frac{C}{\lambda}\sum_j \int_{B_j} \left|h_j(y)\right| \sum_{k=1}^{\infty} \frac{k2^{-\epsilon_1 k}}{\mu(2^k B_j)} |b_{2^{k+1}B_j}-b_{B_j}| \int_{2^k B_j} \omega(x)\,d\mu(x)\,d\mu(y)$$
$$\leqslant \frac{C}{\lambda}\sum_j \int_{B_j} \left|h_j(y)\right| \sum_{k=1}^{\infty} \frac{k2^{-\epsilon_1 k}}{\mu(2^k B_j)} \int_{2^k B_j} \omega(x)\,d\mu(x)\,d\mu(y)$$
$$\leqslant \frac{C}{\lambda}\sum_j \int_{B_j} \left|h_j(y)\right|\omega(y)\,d\mu(y) \leqslant \frac{C}{\lambda}\|f\|_{L^1(\omega)}.$$

To estimate II, applying Theorem 1.1, we know that H is bounded from $L^1(\omega)$ to $L^{1,\infty}(\omega)$, we obtain

$$II = \omega\left(\left\{y \in \mathbb{R}^n \setminus \bar{\Omega}\colon \left|H\left(\sum_j \big(b(x)-b_{B_j}\big)h_j\right)(x)\right| > \lambda/4\right\}\right)$$
$$\leqslant \frac{C}{\lambda}\sum_j \int_{\mathbb{R}^n} \left|b(x)-b_{B_j}\right|\left|h_j(x)\right|\omega(x)\,d\mu(x)$$
$$= \frac{C}{\lambda}\sum_j \int_{\mathbb{R}^n} \left|b(x)-b_{B_j}\right|\left|h_j(x)\right|\omega(x)\,d\mu(x)$$
$$\leqslant \frac{C}{\lambda}\sum_j \int_{B_j} \left|b(x)-b_{B_j}\right|\left|f(x)\right|\omega(x)\,d\mu(x)$$
$$+\frac{C}{\lambda}\sum_j \frac{1}{\mu(B_j)} \int_{B_j} \left|f(y)\right|d\mu(y) \int_{B_j} \left|b(x)-b_{B_j}\right|\omega(x)\,d\mu(x)$$
$$\leqslant \frac{C}{\lambda}\sum_j \int_{\mathbb{R}^n} \left|b(x)-b_{B_j}\right|\left|f(x)\right|\omega(x)\,d\mu(x) + \frac{C}{\lambda}\sum_j \inf_{x\in B_j}\omega(x) \int_{B_j} \left|f(y)\right|d\mu(y)$$
$$\leqslant \frac{C}{\lambda}\sum_j \omega(B_j)\|b-b_{B_j}\|_{\exp L_\omega, B_j}\|f\|_{L\log L_\omega} + \frac{C}{\lambda}\sum_j \int_{B_j} \left|f(y)\right|\omega(y)\,d\mu(y)$$
$$\leqslant \frac{C}{\lambda}\sum_j \omega(B_j)\inf\left\{t>0\colon t+\frac{t}{\omega(B_j)}\int_{B_j} \frac{|f(y)|}{t}\log\left(2+\frac{|f(y)|}{t}\right)\omega(y)\,d\mu(y)\right\}$$
$$+\frac{C}{\lambda}\int_{\mathbb{R}^n} \left|f(y)\right|\omega(y)\,d\mu(y)$$
$$\leqslant C\sum_j \int_{B_j} \frac{|f(y)|}{\lambda}\log\left(2+\frac{|f(y)|}{\lambda}\right)\omega(y)\,d\mu(y) + \frac{C}{\lambda}\int_{\mathbb{R}^n} \left|f(y)\right|\omega(y)\,d\mu(y)$$
$$\leqslant C\int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\log\left(2+\frac{|f(y)|}{\lambda}\right)\omega(y)\,d\mu(y).$$

In the above equalities, we used the following facts:

$$\|f\|_{L\log L_\omega,B}=\inf\left\{\lambda>0\colon\ \frac{1}{\omega(B)}\int\limits_B\frac{|f(y)|}{\lambda}\log\left(2+\frac{|f(y)|}{\lambda}\right)\omega(y)\,dy\leqslant 10\right\}$$

and

$$\|f\|_{\exp L_\omega,B}=\inf\left\{\lambda>0\colon\ \frac{1}{\omega(B)}\int\limits_B\exp\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\,dy\leqslant 10\right\},$$

where $\omega(B)=\int_B\omega(y)\,d\mu(y)$. The generalized Hölder inequality

$$\frac{1}{\omega(B)}\int\limits_B\bigl|f(y)h(y)\bigr|\omega(y)\,d\mu(y)\leqslant C\|f\|_{L\log L_\omega,B}\|h\|_{\exp L_\omega,B}.$$

Since $\|b\|_{BMO_{\mathcal{F}}}=1$, then $\mu(\{x\in B\colon\ |b(x)-b_B|>\lambda\})\leqslant C\exp(-c_1\lambda)\mu(B)$. Since $\omega\in A_1$, hence there exists $\delta>0$ such that

$$\omega\bigl(\bigl\{x\in B\colon\ |b(x)-b_B|>\lambda\bigr\}\bigr)\leqslant C\exp(-\delta c_1\lambda)\omega(B),$$

which implies that $\|b-b_{B_j}\|_{\exp L_\omega,B_j}\leqslant C$. Thus, Theorem 1.3 is proved. □

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